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A Galerkin Method with Smoothing

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This paper presents a modified version of the Galerkin method in which the original bilinear form and the corresponding linear functional are perturbed by means of a smoothing parameter. Although, as Cea's lemma shows, it is not possible to improve the rate of convergence, we prove that our scheme provides a smaller error bound than the usual Galerkin solution. Also, a procedure to obtain an approximation of the projection of the exact solution, which gives a better rate of convergence than the Galerkin solution, is suggested. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and corresponding norm $\|\cdot\|_H$.

Let $B: H \times H \to \mathbf{R}$ be a continuous H-elliptic bilinear form. Let M and α denote the constant of continuity and the constant of coerciveness of B, respectively, i.e.,

$$|B(v, w)| \leqslant M \|v\|_H \|w\|_H, \qquad \forall v, w \in H \tag{1.1}$$

and

$$\alpha \|v\|_H^2 \leqslant B(v, v), \qquad \forall v \in H. \tag{1.2}$$

Also, let $f: H \to \mathbf{R}$ be a continuous linear form. We are interested in finding $u \in H$ such that

$$B(u, v) = f(v), \qquad \forall v \in H. \tag{1.3}$$

It is well known that under the above assumptions, the Lax–Milgram

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lemma (cf. [8, Theorem 1.1.3]) provides the existence and uniqueness for the solution of (1.3).

Now, let $\{S^h\}_{h>0}$ be a parametric family of finite dimensional subspaces of H. We consider the problem: Find $u^h \in S^h$ such that

$$B(u^h, v) = f(v), \qquad \forall v \in S^h. \tag{1.4}$$

It is easily seen that (1.4) has one and only one solution. Moreover, the Cea's lemma (cf. [8, Theorem 2.4.1]) gives the following error estimate

$$\|u - u^h\|_H \leqslant \left(\frac{M}{\alpha}\right) \min_{v \in S^h} \|u - v\|_H \tag{1.5}$$

which shows that the problem of estimating the error $\|u - u^h\|_H$ is reduced to a problem in approximation theory.

An important remark is that if B is symmetric, then an equivalent formulation to (1.4) is given by

$$\|u - u^h\|_E = \min_{v \in S^h} \|u - v\|_E, \tag{1.6}$$

where $\|\cdot\|_E$ is the energy norm induced by the inner product B. Furthermore, by using (1.6), we obtain instead of (1.5)

$$\|u - u^h\|_H \leqslant \left(\frac{M}{\alpha}\right)^{1/2} \min_{v \in S^h} \|u - v\|_H. \tag{1.7}$$

Since (1.1) and (1.2) imply $\alpha \leqslant M$, we can see that (1.7) is better than (1.5).

2. The Smoothing Solution

Let u and u^h be the solutions of (1.3) and (1.4), respectively. Let $g \in H'$ so that

$$g(v) = \langle u, v \rangle_H, \qquad \forall v \in H. \tag{2.1}$$

By following the same ideas of [9], for a given parameter $\psi > 0$, we consider the problem: Find $u_h(\psi) \in S^h$ such that

$$B(u_h(\psi), v) + \psi < u_h(\psi), v >_H = f(v) + \psi g(v), \qquad \forall v \in S^h. \tag{2.2}$$

We have the following result

PROPOSITION 2.1. *For any $\psi > 0$, there exists a unique $u_h(\psi)$ solution of* (2.2). *Moreover*

$$\|u - u_h(\psi)\|_H \leqslant \left(\frac{M+\psi}{\alpha+\psi}\right) \min_{v \in S^h} \|u - v\|_H. \tag{2.3}$$

Proof. Let $a: H \times H \to \mathbf{R}$ be the bilinear form

$$a(v, w) = B(v, w) + \psi \langle v, w \rangle_H$$

and let $F: H \to \mathbf{R}$ be the linear functional

$$F(v) = f(v) + \psi g(v).$$

We have clearly

$$|a(v, w)| \leqslant |B(v, w)| + \psi \, |\langle v, w \rangle_H| \leqslant (M + \psi) \, \|v\|_H \, \|w\|_H$$

and

$$a(v, v) = B(v, v) + \psi \langle v, v \rangle_H \geqslant (\alpha + \psi) \, \|v\|_H^2.$$

Since $a(u, v) = F(v)$ for all $v \in H$, a direct application of Lax–Milgram and Cea's lemmas complete the proof. ∎

The unique $u_h(\psi) \in S^h$ solution of (2.2) will be called *the smoothing Galerkin solution with parameter* ψ.

Now, let P_h be the projection of H on S^h, i.e.,

$$\begin{gathered} P_h : H \to S^h \\ v \to P_h v := \tilde{v}_h \\ \|v - \tilde{v}_h\|_H = \min_{w \in S^h} \|v - w\|_H. \end{gathered} \tag{2.4}$$

Then, we obtain

PROPOSITION 2.2. *For every $\psi > 0$, $u_h(\psi)$ satisfies*

$$B(u_h(\psi), v) + \psi \langle u_h(\psi), v \rangle_H = f(v) + \psi \tilde{g}(v), \qquad \forall v \in S^h, \tag{2.5}$$

where

$$\tilde{g}(v) = \langle \tilde{u}_h, v \rangle_H, \qquad \forall v \in S^h. \tag{2.6}$$

Proof. It follows easily from (2.2) and the fact that $\langle u - \tilde{u}_h, v \rangle_H = 0$, $\forall v \in S^h$. ∎

Since M is necessarily larger than α, we deduce that for any positive ψ the constant term in (2.3) is smaller than that of (1.5). In other words, the approximation to u given by $u_h(\psi)$ provides a better error bound than that given by the usual Galerkin solution u^h. In order to make this fact even more clear, we note that, although $(M+\psi)/(\alpha+\psi)$ is always greater than 1, we can make this quotient arbitrarily close to 1 by choosing ψ appropriately. It is easily seen that for any $\delta > 0$

$$\psi \geqslant \frac{M - \alpha(\delta+1)}{\delta}$$

is a necessary and sufficient condition to obtain

$$\frac{M+\psi}{\alpha+\psi} \leqslant 1+\delta.$$

For instance, $\delta = 1$ requires $\psi \geqslant M - 2\alpha$ which gives

$$\frac{M+\psi}{\alpha+\psi} \leqslant 2 = O(1).$$

In this case, we can write

$$\|u - u_h(\psi)\|_H \leqslant 2 \min_{v \in S^h} \|u - v\|_H,$$

i.e., the error in the approximation of $u_h(\psi)$ to u is at most two times the smallest error guaranteed by the subspace S^h.

The importance of this is evident in cases in which the constant of coerciveness α is small. Under this situation it is very clear, at least in terms of a priori estimates, the superiority of this smoothing scheme versus the usual Galerkin scheme which, as shown in (1.5), gives an approximation of $O(1/\alpha)$.

Nevertheless, as we can see from (2.1)–(2.2) and (2.5)–(2.6), it is impossible to obtain explicitly $u_h(\psi)$, unless we know either the exact solution u or its projection $\tilde{u}_h$. Therefore, in Section 3 we will address the question of how to choose an approximation for $u_h(\psi)$.

From now on, $\{e_1, ..., e_N\}$ will denote a basis of the subspace S^h.

PROPOSITION 2.3. *Let C and F be the stiffness matrix and the load vector, respectively, associated with the Galerkin solution u^h, i.e.,*

$$C = (c_{ij})_{N \times N}; \qquad c_{ij} := B(e_i, e_j)$$

$$F = (f_i)_{N \times 1}; \qquad f_i := f(e_i).$$

Let $E = (e_{ij})_{N \times N}$ *and* $K = (k_j)_{N \times 1}$ *be defined by*

$$e_{ij} := \langle e_i, e_j \rangle_H, \qquad k_j := \langle \tilde{u}_h, e_j \rangle_H = \langle u, e_j \rangle_H. \tag{2.7}$$

Then, for any $\psi > 0$, $u_h(\psi)$ *is given by*

$$u_h(\psi) = \sum_{j=1}^{N} z_j(\psi)\, e_j, \tag{2.8}$$

where $Z(\psi) := (z_1(\psi), ..., z_N(\psi))^T$ *is obtained from the system*

$$(C + \psi E)\, Z(\psi) = F + \psi K. \tag{2.9}$$

Proof. Since $\{e_1, ..., e_N\}$ is a basis of S^h, Eq. (2.5) is equivalent to

$$B(u_h(\psi), e_j) + \psi \langle u_h(\psi), e_j \rangle_H = f(e_j) + \psi \tilde{g}(e_j), \qquad \forall j = 1, ..., N.$$

By setting $u_h(\psi) = \sum_{i=1}^{N} z_i(\psi)\, e_i$ in the above expression, we obtain clearly (2.9). ∎

3. The Approximate Smoothing Solution

The only one difficulty in using (2.5) to obtain the smoothing solution $u_h(\psi)$ is the evaluation of $\tilde{g}$. Hence, we suggest computing an approximation of $u_h(\psi)$ by considering a known functional $\hat{g}$ such that $\|\tilde{g} - \hat{g}\|_{H'}$ is sufficiently small.

In order to do this, we assume that we have at our disposal an approximation $\hat{u} \in S^h$ to $\tilde{u}_h$.

With this additional information we define

$$\begin{aligned} \hat{g}&: H \to \mathbf{R} \\ v &\to \hat{g}(v) = \langle \hat{u}, v \rangle_H \end{aligned} \tag{3.1}$$

and consider the problem: Find $\hat{u}_h(\psi) \in S^h$ such that

$$B(\hat{u}_h(\psi), v) + \psi \langle \hat{u}_h(\psi), v \rangle = f(v) + \psi \hat{g}(v), \qquad \forall v \in S^h. \tag{3.2}$$

Given $\hat{u} \in S^h$ fixed, the unique $\hat{u}_h(\psi) \in S^h$ solution of (3.2) will be called *the approximate smoothing Galerkin solution with parameter* ψ. It is worth remarking that $\hat{u}_h(\psi)$ is obtained from the system (2.9) with $\hat{k}_j = \langle \hat{u}, e_j \rangle_H$ instead of k_j.

The following result provides the corresponding error bounds for the differences $u_h(\psi) - \hat{u}_h(\psi)$ and $u - \hat{u}_h(\psi)$.

PROPOSITION 3.1. *Let $\hat{u} \in S^h$ be an approximation to $\tilde{u}_h$. Then, for any $\psi > 0$, we get*

$$\|u_h(\psi) - \hat{u}_h(\psi)\|_H \leqslant \left(\frac{\psi}{\alpha + \psi}\right) \|\hat{u} - \tilde{u}_h\|_H \tag{3.3}$$

and

$$\|u - \hat{u}_h(\psi)\|_H \leqslant \left(\frac{M + \psi}{\alpha + \psi}\right) \|u - \tilde{u}_h\|_H + \left(\frac{\psi}{\alpha + \psi}\right) \|\hat{u} - \tilde{u}_h\|_H. \tag{3.4}$$

Proof. From (2.5) and (3.2) we obtain

$$B(\hat{u}_h(\psi) - u_h(\psi), v) + \psi \langle \hat{u}_h(\psi) - u_h(\psi), v \rangle_H = \psi(\hat{g}(v) - \tilde{g}(v)) \tag{3.5}$$

for all $v \in S^h$.

In particular, for $v = \hat{u}_h(\psi) - u_h(\psi)$, (3.5) is transformed in

$$\begin{aligned} B(\hat{u}_h(\psi) - u_h(\psi), \hat{u}_h(\psi) - u_h(\psi)) + \psi \ \|\hat{u}_h(\psi) - u_h(\psi)\|_H^2 \\ = \psi \langle \hat{u} - \tilde{u}_h, \hat{u}_h(\psi) - u_h(\psi) \rangle_H. \end{aligned} \tag{3.6}$$

By using the H-ellipticity of B and the Cauchy–Schwarz inequality in (3.6), we deduce

$$(\alpha + \psi) \ \|\hat{u}_h(\psi) - u_h(\psi)\|_H^2 \leqslant \psi \ \|\hat{u} - \tilde{u}_h\|_H \ \|\hat{u}_h(\psi) - u_h(\psi)\|_H$$

and hence (3.3).

Finally, (3.4) is a consequence of (3.3), Proposition 2.1, and the triangle inequality. ▌

It is clear that as a first choice of $\hat{u}$ we could utilize the usual Galerkin solution u^h. However, in this case one obtains from (3.2) that $\hat{u}_h(\psi) = u^h$, $\forall \psi > 0$. Therefore, in Section 5 we present an alternative procedure which improves this approximation.

On the other hand, the approximate scheme (3.2) yields a modified h-version of the finite element method in which the following sequence of discrete problems is considered

$$B(\hat{u}_1, v) = f(v), \qquad \forall v \in S^{h_1} \tag{3.7}$$

and for $j \geqslant 2$

$$\begin{aligned} B(\hat{u}_j(\psi), v) + \psi \langle \hat{u}_j(\psi), v \rangle_H \\ = f(v) + \psi \langle \hat{u}_{j-1}(\psi), v \rangle_H, \qquad \forall v \in S^{h_j}. \end{aligned} \tag{3.8}$$

For a given parameter $\psi > 0$, the set $\{\hat{u}_j(\psi)\}_{j \in \mathbb{N}}$ satisfying (3.7)–(3.8) constitutes the approximating sequence to the exact solution. Here, as usual, $S^{h_{j+1}} \subseteq S^{h_j}$ for all $j \in \mathbb{N}$, and $\{h_j\}_{j \in \mathbb{N}}$ is a decreasing sequence of positive numbers with limit zero.

In the same way, if instead of $\{S^{h_j}\}_{j \in \mathbb{N}}$ we consider a sequence of finite dimensional subspaces $\{S^{p_j}\}_{j \in \mathbb{N}}$ in which p_j denotes an increasing degree of polynomial approximation, then (3.7)–(3.8) can be interpreted as a modified p-version of the finite element method (see [2, 4, 10]).

4. The Symmetric Case

In addition to the above hypotheses on the bilinear form, let us suppose here that B is symmetric. Then, it is easily seen that the smoothing Galerkin solution can also be characterized by the following minimization problem

$$\|u - u_h(\psi)\|_E^2 + \psi \, \|u - u_h(\psi)\|_H^2 = \min_{v \in S^h} \{\|u - v\|_E^2 + \psi \| u - v\|_H^2\}, \quad (4.1)$$

where, as usual, $\|\cdot\|_E$ is the energy norm induced by B.

In this case, we obtain the following result

Proposition 4.1. *For any $\psi > 0$, we have*

$$\|u - u_h(\psi)\|_H^2 \leqslant \|u - u^h\|_H^2 - V(\psi), \quad (4.2)$$

where

$$V(\psi) = \frac{\|u_h(\psi) - u^h\|_E^2}{\psi}. \quad (4.3)$$

Proof. By setting $v = u^h$ in (4.1) we deduce clearly

$$\|u - u_h(\psi)\|_E^2 + \psi \, \|u - u_h(\psi)\|_H^2 \leqslant \|u - u^h\|_E^2 + \psi \, \|u - u^h\|_H^2, \quad (4.4)$$

that is,

$$\|u - u_h(\psi)\|_H^2 \leqslant \|u - u^h\|_H^2 - \frac{1}{\psi} \{\|u - u_h(\psi)\|_E^2 - \|u - u^h\|_E^2\}. \quad (4.5)$$

Now, since $B(u - u^h, v) = 0$, for all $v \in S^h$, we get

$$\|u - u_h(\psi)\|_E^2 = \|u - u^h\|_E^2 + \|u^h - u_h(\psi)\|_E^2$$

and hence

$$\|u-u_h(\psi)\|_E^2-\|u-u^h\|_E^2=\|u^h-u_h(\psi)\|_E^2 \tag{4.6}$$

Thus, (4.2) follows from (4.5) and (4.6). ∎

It is important to remark that in the symmetric case, the error bounds (2.3) and (3.4) are improved by

$$\|u-u_h(\psi)\|_H \leqslant \left(\frac{M+\psi}{\alpha+\psi}\right)^{1/2} \|u-\tilde{u}_h\|_H$$

and

$$\|u-\hat{u}_h(\psi)\|_H \leqslant \left(\frac{M+\psi}{\alpha+\psi}\right)^{1/2} \|u-\tilde{u}_h\|_H + \left(\frac{\psi}{\alpha+\psi}\right) \|\hat{u}-\tilde{u}_h\|_H,$$

respectively.

The function V in Proposition 4.1 will be called the optimality function. Moreover, the following proposition shows the existence of an optimal parameter.

PROPOSITION 4.2. *There exists $\psi_0>0$ so that*

$$V(\psi_0)=\sup_{\psi\in(0,+\infty)} V(\psi).$$

Proof. First of all, let us note that the Galerkin solution u^h coincides with $u_h(0)$. So, from Proposition 2.3 we can write

$$u^h=\sum_{j=1}^N z_j(0)\,e_j \qquad \text{and} \qquad u_h(\psi)=\sum_{j=1}^N z_j(\psi)\,e_j, \tag{4.7}$$

where

$$CZ(0)=F \qquad \text{and} \qquad (C+\psi E)\,Z(\psi)=F+\psi K.$$

After some computations we obtain

$$Z(\psi)-Z(0)=\psi[C+\psi E]^{-1}[K-EZ(0)]. \tag{4.8}$$

On the other hand, by using (4.7) in (4.3) we have

$$V(\psi)=\frac{1}{\psi}\sum_{i=1}^N\sum_{j=1}^N (z_i(\psi)-z_i(0))(z_j(\psi)-z_j(0))\,B(e_i,e_j),$$

i.e.,

$$V(\psi)=\frac{1}{\psi}(Z(\psi)-Z(0))^T C(Z(\psi)-Z(0)).$$

By substituting (4.8) into the above expression we obtain

$$V(\psi)=\psi[K-EZ(0)]^T[C+\psi E]^{-1}C[C+\psi E]^{-1}[K-EZ(0)] \tag{4.9}$$

or equivalently

$$V(\psi)=\frac{1}{\psi}[K-EZ(0)]^T\left[E+\frac{1}{\psi}C\right]^{-1}C\left[E+\frac{1}{\psi}C\right]^{-1}[K-EZ(0)]. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\lim_{\psi\to 0} V(\psi)=\lim_{\psi\to +\infty} V(\psi)=0. \tag{4.11}$$

Therefore, for $\varepsilon = V(1)$, there exist positive constants δ and R such that

$$V(\psi)<V(1), \qquad \forall\psi\in(0,\delta)\cup(R,+\infty).$$

Hence, since V is continuous on $(0,+\infty)$, we can put

$$\sup_{\psi\in(0,+\infty)} V(\psi)=\max_{\psi\in[\delta,R]} V(\psi)$$

which completes the proof. ∎

5. An Estimate of the Projection

Let us suppose that the approximation properties of the subspace S^h are characterized by the relation

$$\|v-\tilde{v}_h\|_H \leqslant C_1 h^N G(v), \qquad \forall v\in\hat{H}\subset H, \tag{5.1}$$

where C_1 is a positive constant independent of h and v, G is a function depending only on v, usually a norm on the subspace $\hat{H}$ of H, and N is a positive integer (see [1; 8, Theorem 3.2.1]). Also, $\tilde{v}_h := P_h v$, P_h being the projection already defined in (2.4).

We denote $\tilde{G}(v) := C_1 G(v)$, for all $v \in \hat{H}$.

According to (5.1) and (1.5), the Galerkin solution u^h satisfies

$$\|u - u^h\|_H \leqslant \frac{M}{\alpha} h^N \tilde{G}(u) \tag{5.2}$$

and therefore

$$\|\tilde{u}_h - u^h\|_H \leqslant \frac{M}{\alpha} h^N \tilde{G}(u). \tag{5.3}$$

Here, we have assumed implicitly that $u \in \hat{H}$.

Now, as it was remarked in Section 2, the application of the smoothing scheme should provide better results than the usual Galerkin solution. However, in order to apply that scheme, we need a good estimate of $\tilde{u}_h$. Thus, in this section we give a procedure to obtain a better approximation than u^h for the projection of the solution u. In this way, we will be able to use successfully the approximate smoothing scheme proposed in Section 3.

We assume that by using either a numerical technique or an analytic method, we have obtained an approximation $u^a \in \hat{H}$ of u which is not in S^h. It is interesting to point out that this kind of assumption arises for instance in the multigrid context (see [6, 12, 13]) where the correction process at one level (or subspace S^h) requires of the previously computed solution at the next higher level. Also, we should mention that in the case of boundary layer problems the use of asymptotic expansions constitutes a systematic procedure to construct approximate analytical solutions (see [7, 11] where this approach has been used).

We now assume that there exist $m \geqslant 1$ and a positive constant C_2 independent of h but that may depend on u, such that

$$G(u - u^a) \leqslant C_2 h^m G(u). \tag{5.4}$$

Then, we consider the problem: Find $\hat{u} \in S^h$ such that

$$B(\hat{u}, v) = f(v) + B(\tilde{u}_h^a - u^a, v), \qquad \forall v \in S^h. \tag{5.5}$$

Here, $\tilde{u}_h^a := P_h u^a$ is the projection on S^h of the approximate solution u^a. It is important to remark that our scheme (5.5) is similar to the asymptotic one presented in [7], but with a different approach. As a matter of fact, Bar–Yoseph and Israeli propose in [7] the same variational formulation, but instead of the projection P_h they use the interpolation operator (cf. [8]).

PROPOSITION 5.1. *Let $\hat{u}$ be the solution of the scheme* (5.5). *Then, we obtain*

$$\|\tilde{u}_h - \hat{u}\|_H \leqslant C_2 \frac{M}{\alpha} h^{N+m} \tilde{G}(u), \tag{5.6}$$

where C_2 is the constant of (5.4).

Proof. Let us denote $e^a := u - u^a$. For any v in S^h we have

$$B(\tilde{u}_h - \hat{u}, v) = B(\tilde{u}_h, v) - f(v) - B(\tilde{u}_h^a - u^a, v).$$

Since $B(u, v) = f(v)$, we can write

$$B(\tilde{u}_h - \hat{u}, v) = B(\tilde{u}_h, v) - B(u, v) - B(\tilde{u}_h^a, v) + B(u^a, v)$$

that is

$$B(\tilde{u}_h - \hat{u}, v) = B(\tilde{u}_h - \tilde{u}_h^a, v) - B(u - u^a, v).$$

Now, the linearity of P_h implies

$$P_h e^a := \tilde{e}_h^a = \tilde{u}_h - \tilde{u}_h^a.$$

It follows that

$$B(\tilde{u}_h - \hat{u}, v) = B(\tilde{e}_h^a - e^a, v), \qquad \forall v \in S^h. \tag{5.7}$$

In particular, $v = \tilde{u}_h - \hat{u}$ in (5.7) gives

$$B(\tilde{u}_h - \hat{u}, \tilde{u}_h - \hat{u}) = B(\tilde{e}_h^a - e^a, \tilde{u}_h - \hat{u}). \tag{5.8}$$

By using the H-ellipticity and continuity of B in (5.8), we deduce

$$\|\tilde{u}_h - \hat{u}\|_H \leqslant \frac{M}{\alpha} \|\tilde{e}_h^a - e^a\|_H. \tag{5.9}$$

But, according to (5.1), we have

$$\|e^a - \tilde{e}_h^a\|_H \leqslant h^N \tilde{G}(e^a). \tag{5.10}$$

Finally, by combining (5.4), (5.9), and (5.10) we complete the proof. ∎

By comparison of (5.3) and (5.6), we see that the scheme (5.5) provides an approximation $\hat{u}$ for the projection of u which improves in a factor of order h^m the approximation given by the Galerkin solution.

Our solution $\hat{u}_h(\psi)$, obtained from (3.2) with $\hat{u}$ given by (5.5), will again be called *the approximate smoothing Galerkin solution.*

Proposition 5.1 predicts an improvement by a factor of h^m of the a priori error estimate of u^h with respect to the projection $\tilde{u}_h$. However, we cannot in general predict an improvement of the error itself. Now, the error bound given by Proposition 5.1 is based on the existence of u^a in the complement of S^h such that it satisfies (5.4). Hence, in the particular case of the finite element method, a useful suggestion is to combine this approach with an adaptive refinement technique (see [3, 5, 14]) in which a nested sequence of meshes is created. More precisely, one may consider u^a as the finite element solution on either the same mesh associated to S^h or a coarser mesh, but where a higher order of approximation is used (see [2, 4]). In this way, if the exact solution u is smooth enough then the error bound (5.4) would be easily proved by using the interpolation theory of Sobolev spaces (see [1; 8, Chapter 3]). Further details and some numerical experiments will be available in [10].

As a consequence of Proposition 3.1 and Proposition 5.1, we state the following theorem.

THEOREM 5.1. *Let u^a be an approximation of u satisfying* (5.4). *Suppose that the subspace S^h satisfies the approximation property* (5.1). *Let $\hat{u}$ be the solution of the scheme* (5.5). *Then, for any $\psi > 0$, the approximate smoothing Galerkin solution satisfies the error bound*

$$\|u - \hat{u}_h(\psi)\|_H \leqslant \left\{\frac{M + \psi(1 + C_2 M h^m/\alpha)}{\alpha + \psi}\right\} h^N \tilde{G}(u) \tag{5.11}$$

or equivalently

$$\|u - \hat{u}_h(\psi)\|_H \leqslant \left\{\frac{M + C_3\psi}{\alpha + \psi}\right\} h^N \tilde{G}(u), \tag{5.12}$$

where $C_3 = O(1)$.

Proof. It follows from (3.4), (5.1) and (5.6). ∎

We remark finally that for h sufficiently small the constant term in our estimate (5.12)

$$\frac{M + C_3\psi}{\alpha + \psi}$$

is bounded below by C_3, which is the limiting case as $\psi \to +\infty$. We believe, however, that a very large value of ψ is not practical numerically because of the loss of precision which would result. Some numerical tests on this matter and some practical criteria for choosing ψ will be presented in [10].

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